

# Massive Corrections to Entanglement in Minimal $E_8$ Toda Field Theory

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In this letter we study the exponentially decaying corrections to saturation of the second Rényi entropy of one interval of length  $\ell$  in minimal  $E_8$  Toda field theory. It has been known for some time that the entanglement entropy of a massive quantum field theory in 1+1 dimensions saturates to a constant value for  $m_1\ell \gg 1$  where  $m_1$  is the mass of the lightest particle in the spectrum. Subsequently, results by Cardy, Castro-Alvaredo and Doyon have shown that there are exponentially decaying corrections to this behaviour which are characterized by Bessel functions with arguments proportional to  $m_1\ell$ . For the von Neumann entropy the leading correction to saturation takes the precise universal form  $-\frac{1}{8}K_0(2m_1\ell)$  whereas for the Rényi entropies leading corrections which are proportional to  $K_0(m_1\ell)$  are expected. Recent numerical work by Pálmai for the second Rényi entropy of minimal  $E_8$  Toda has found next-to-leading order corrections decaying as  $e^{-2m_1\ell}$  rather than the expected decay as  $e^{-m_1\ell}$ . In this paper we investigate the origin of this result and show how it might be reconciled with analytical results based on the computation of correlators of branch point twist fields.

**Keywords:** Integrable Quantum Field Theory, Entanglement Entropy, Form Factors, Twist Fields

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# 1 Introduction

## 1.1 Entanglement Entropy

Measures of entanglement, such as the entanglement entropy (EE), have attracted much attention in recent years, particularly in the context of one-dimensional many body quantum systems (see e.g. review articles in the special issue [1]). Among such systems, those enjoying conformal invariance in the scaling limit are of particular interest as they provide a theoretical and universal description of critical phenomena. In their seminal work Calabrese and Cardy [2] used principles of Conformal Field Theory (CFT) to study the (EE) [3] of quantum critical systems. Their results generalised previous work [4], provided theoretical support for numerical observations in critical quantum spin chains [5] and highlighted the fact that the EE encodes universal information about quantum critical points, such as the central charge of the corresponding CFT. This information may be extracted numerically in a very efficient way, typically by employing Density Matrix Renormalization Group methods [6], and this has provided one of the main motivations to investigate measures of entanglement in critical and near-critical systems. From a mathematical physics viewpoint (the one taken in this paper) the investigation of the EE is driven by interest in developing a better (if possible, analytical) understanding of the universal properties of the ground state of extended many body quantum systems.

The EE is a measure of the amount of quantum entanglement, in a pure quantum state, between the degrees of freedom associated to two sets of independent observables whose union is complete on the Hilbert space. In the present paper, the two sets of observables correspond to the local observables in two complementary connected regions,  $A$  and  $B$ , of a 1+1-dimensional massive quantum field theory (QFT), and we will consider only the case where the quantum state is the ground state. Let  $|\Psi\rangle$  be such a ground state. Consider a space bi-partition of the theory as sketched in Figure 1. Then the EE associated to region  $A$  may be expressed as  $S(\ell) = -\text{Tr}(\rho_A \log \rho_A)$  where  $\rho_A = \text{Tr}_B(|\Psi\rangle\langle\Psi|)$  is the reduced density matrix associated to subsystem  $A$  and  $\ell$  is the subsystem's length.

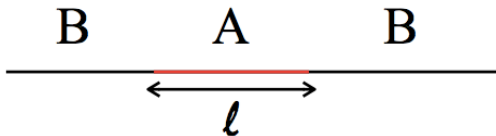


Figure 1: Typical bi-partition for the EE of one interval.

The EE defined above is also known as von Neumann entropy. Alternative, related definitions of the entanglement entropy have been proposed which are also frequently studied. A set of popular measures is provided by the Rényi entropies which are defined as

$$S_n(\ell) = \frac{\log \text{Tr} \rho_A^n}{1-n}, \quad (1)$$

and have the property  $\lim_{n \rightarrow 1} S_n(\ell) = S(\ell)$ . In this paper we will consider the case  $n = 2$  where  $S_2(\ell) = -\log \text{Tr} \rho_A^2$ . We will refer to this quantity as the *second Rényi entropy*. We choose this particular value in order to compare with results obtained in [7] for the same quantity by a different method.

As mentioned earlier, at quantum critical points, the scaling limit of the EE has been widely studied in CFT [8, 4, 5, 9, 2, 10] and in lattice realizations of critical systems such as quantum spin chains [11, 12, 13, 14, 15, 16, 17] and lattice models [18, 19, 20]. In particular, the combination

of a geometric description, Riemann uniformization techniques and standard expressions for CFT partition functions is very fruitful. Recently [21], this was generalized to non-unitary CFT and to the EE of excited states [22, 23], where general formulae were obtained using also such techniques. Near critical points, the scaling limit is instead described by massive quantum field theory (QFT), and geometric techniques relying on conformal mappings break down. So far the most powerful way of studying the EE in massive QFT is by using an approach based on local *branch-point twist fields* [24, 25, 26]. This approach is very fruitful and complete as it allows both for numerical and analytical computations. In this context, the second Rényi entropy may be defined as:

$$S_2(\ell) = -\log \left( \epsilon^{4\Delta_2} \langle \mathcal{T}(0) \tilde{\mathcal{T}}(\ell) \rangle_2 \right), \quad (2)$$

where  $\mathcal{T}$  and  $\tilde{\mathcal{T}} := \mathcal{T}^\dagger$  are the branch point twist fields,

$$\Delta_n = \frac{c}{24} \left( n - \frac{1}{n} \right), \quad (3)$$

is their conformal dimension (at criticality) [27, 28, 2] as a function of the central charge  $c$ , and  $\epsilon$  is a non-universal short-distance cut-off. The expression  $\langle \mathcal{T}(0) \tilde{\mathcal{T}}(\ell) \rangle_2$  above denotes the two-point function in the ground state for  $n = 2$ . An important subtlety is that branch point twist fields are local fields in a new QFT which is constructed as  $n$  non-interacting copies (in this case 2) of the original QFT. In this context, they are interpreted as symmetry fields associated to the cyclic permutation symmetry of the “replica” theory.

In this paper we aim to compare results based on a branch point twist field approach to recent numerical results by Tamás Pálmai [7] for the quantity (2). In [7] a new approach to the computation of the second Rényi entropy in massive integrable QFT was proposed which is based on the use of the Truncated Conformal Space Approach (TCSA) first proposed by Yurov and Zamolodchikov in [29]. The TCSA is based on Zamolodchikov’s view of massive integrable models as massive perturbations of CFT [30]. It exploits the rich structure of the Hilbert space of CFT, perturbs and truncates the latter and then diagonalizes the “truncated” Hamiltonian. This provides a very successful way to access the low energy spectrum of massive integrable QFT with (a priori) any desired level of accuracy. The work [7] showed for the first time that TCSA can also be employed to access the quantity  $\text{Tr} \rho_A^2$ , and so may be applied to the study of measures of entanglement. This is a very interesting development which complements and enhances the existing twist field approach for massive QFTs.

## 1.2 The Model

In this paper we consider an integrable massive QFT sometimes referred to as the critical Ising model in a magnetic field (IMMF) and also known as the minimal  $E_8$  Toda field theory. In the spirit of Zamolodchikov’s work [30], the theory can be described as a massive perturbation of the conformal Ising model whose operator content consists of simply three fields: the identity, the energy field  $\epsilon$  and the spin field  $\sigma$ . It is well-known that a massive perturbation by the energy field gives rise to an integrable QFT known as the massive Ising model. This theory has a single particle and the two-body scattering matrix is simply  $S(\theta) = -1$  as a function of the rapidity  $\theta$ . Surprisingly, perturbing with the spin field  $\sigma$  instead gives rise to a much

more complex but still integrable interacting QFT, the IMMF [30, 31]. The theory consists of 8 self-conjugate particles of different masses. All of the particles can also be formed as bound states of two other particles in the spectrum, that is, the corresponding two-body scattering matrices have a rich pole structure in the physical sheet with poles of up to order 12. Following Zamolodchikov's work, a plethora of papers by many authors led to the realization that the IMMF is but a particular case of a much wider family of integrable QFTs known as minimal Toda field theories (a detailed historical account of these findings can be found in [32, 33] and references therein). These in turn are “simplified” versions of another class of models, the Affine Toda field theories (ATFTs), in the sense that the  $S$ -matrices of ATFTs are equal those of minimal Toda theories, up to coupling-dependent multiplicative factors which have no poles in the physical sheet. ATFTs have been studied since a long time and have played a prominent role in the development of the field of integrable field theories [34, 35]. Based on the IMMF

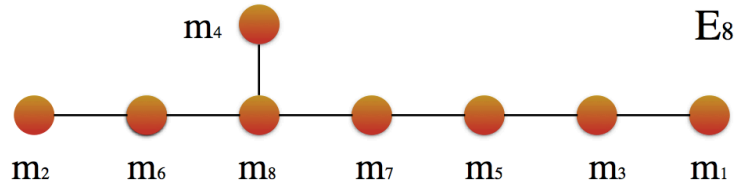


Figure 2:  $E_8$  Dynkin diagram as a representation of the mass spectrum of the IMMF.

example and on extensive work on classical Toda theory it came to be expected that a different theory should exist for each simple Lie algebra and that this Lie algebraic structure would be crucial in the understanding of these models. Subsequently, a lot of work was carried out in order to compute the  $S$ -matrices of ATFTs related to each simple Lie algebra. Much of this work was reviewed and extended in [32], where expressions for the  $S$ -matrices of all ATFTs can be found. More recently a closed universal formula for the  $S$ -matrices of all ATFTs which depends solely on Lie algebraic quantities has been obtained [33]. In this context, the eight particles in the spectrum of the IMMF are in one-to-one correspondence with the simple roots of  $E_8$  (see Fig. 2). Indeed, their values are the Perron-Frobenius eigenvector of the corresponding Cartan matrix. A detailed account of the masses and scattering matrices of the theory can be found for instance in the review [36] and also in [37]. Here we will only report the data that we need for the present paper. We will only require the values of the masses of four lightest particles in the spectrum

$$m_2 = 2m_1 \cos \frac{\pi}{5}, \quad m_3 = 2m_1 \cos \frac{\pi}{30} \quad \text{and} \quad m_4 = 2m_2 \cos \frac{7\pi}{30}. \quad (4)$$

where  $m_1$  is the mass of the lightest particle. We note that the masses of the IMMF satisfy the inequality  $m_i > 2m_1$  for  $i > 3$  whereas  $m_{1,2,3} < 2m_1$ . In the following we will also require some of the two-particle scattering amplitudes as functions of the rapidity variable  $\theta$ . They generally have the structure:

$$S_{ab}(\theta) = \prod_{\alpha \in \mathcal{S}_{ab}} \left[ \frac{\tanh \frac{1}{2}(\theta + i\pi\alpha)}{\tanh \frac{1}{2}(\theta - i\pi\alpha)} \right]^{p_\alpha} \quad \forall \quad a, b = 1, \dots, 8. \quad (5)$$

where  $S_{ab}$  is a known set of integer values which characterises the scattering matrix and  $p_\alpha$  are integer powers which determine the degeneracy of the poles of  $S_{ab}(\theta)$  at  $\theta = i\pi\alpha$ . In particular,

$$S_{11}(\theta) = \frac{\tanh \frac{1}{2} \left( \theta + \frac{2\pi i}{3} \right) \tanh \frac{1}{2} \left( \theta + \frac{2\pi i}{5} \right) \tanh \frac{1}{2} \left( \theta + \frac{i\pi}{15} \right)}{\tanh \frac{1}{2} \left( \theta - \frac{2\pi i}{3} \right) \tanh \frac{1}{2} \left( \theta - \frac{2\pi i}{5} \right) \tanh \frac{1}{2} \left( \theta - \frac{i\pi}{15} \right)}, \quad (6)$$

and

$$S_{12}(\theta) = \frac{\tanh \frac{1}{2} \left( \theta + \frac{4\pi i}{5} \right) \tanh \frac{1}{2} \left( \theta + \frac{3\pi i}{5} \right) \tanh \frac{1}{2} \left( \theta + \frac{7\pi i}{15} \right) \tanh \frac{1}{2} \left( \theta + \frac{4\pi i}{15} \right)}{\tanh \frac{1}{2} \left( \theta - \frac{4\pi i}{5} \right) \tanh \frac{1}{2} \left( \theta - \frac{3\pi i}{5} \right) \tanh \frac{1}{2} \left( \theta - \frac{7\pi i}{15} \right) \tanh \frac{1}{2} \left( \theta - \frac{4\pi i}{15} \right)}, \quad (7)$$

are the only amplitudes we will require in this paper. As we can see,  $S_{11}(\theta)$  has simple poles at  $\frac{2\pi i}{3}$ ,  $\frac{2\pi i}{5}$  and  $\frac{\pi}{15}$  corresponding to the formation of particles 1, 2 and 3, respectively through the scattering processes  $1 + 1 \rightarrow 1$ ,  $1 + 1 \rightarrow 2$  and  $1 + 1 \rightarrow 3$ . Similarly, the scattering matrix  $S_{12}(\theta)$  has four simple poles at  $\frac{4\pi i}{5}$ ,  $\frac{3\pi i}{5}$ ,  $\frac{7\pi i}{15}$  and  $\frac{4\pi i}{15}$  corresponding to the formation of bound states 1, 2, 3 and 4. Associated to these simple poles are the three-point couplings  $\Gamma_{ab}^c$  defined as

$$i(\Gamma_{ab}^c)^2 = \text{Res}_{\theta=i\pi\alpha} S_{ab}(\theta). \quad (8)$$

In particular

$$\begin{aligned} (\Gamma_{11}^1)^2 &= 2\sqrt{15 - 6\sqrt{5}} \cot \frac{\pi}{30} \cot^2 \frac{2\pi}{15}, & (\Gamma_{11}^2)^2 &= (\Gamma_{11}^1)^2 \sqrt{9 + 4\sqrt{5}} \tan \frac{2\pi}{15} \tan \frac{7\pi}{30}, \\ (\Gamma_{11}^3)^2 &= (\Gamma_{11}^2)^2 \frac{\tan \frac{\pi}{30} \tan \frac{\pi}{15}}{\sqrt{9 + 4\sqrt{5}}}, & \Gamma_{12}^1 &= \Gamma_{11}^2, & (\Gamma_{12}^2)^2 &= (\Gamma_{12}^1)^2 (2 + \sqrt{5}) \tan \frac{2\pi}{15} \cot^2 \frac{\pi}{15} \cot \frac{7\pi}{30}, \\ (\Gamma_{12}^3)^2 &= (\Gamma_{12}^1)^2 \cot \frac{\pi}{30} \cot \frac{\pi}{15} \cot \frac{2\pi}{15} \cot \frac{7\pi}{30}, & (\Gamma_{12}^4)^2 &= (\Gamma_{12}^3)^2 \tan \frac{\pi}{30} \tan \frac{2\pi}{15}. \end{aligned} \quad (9)$$

will enter in some of the equations we will see later.

### 1.3 Structure of the Paper

This paper is organised as follows: in section 2 we review the form factor approach for branch point twist fields. We propose expressions for some of the two-particle form factors in the IMMF as well as a set of consistency conditions that allow us to determine also the one-particle form factors of the four lightest particles in the spectrum. In section 3 we explain how the second Rényi entropy may be expressed in terms of twist field form factors and write down an expression including the six leading form factor corrections to its saturation value. We determine the precise coefficients of these corrections by solving the form factor equations proposed in section 2. We confirm the presence of exponentially decaying corrections, led by  $e^{-m_1\ell}$ . In section 4 we compare our results to those obtained in [7] by employing the TCSA approach and discuss their level of agreement. We present our conclusions and outlook in section 5.

## 2 Twist Field Form Factors in the IMMF

### 2.1 Generalities

In 1+1 dimensional integrable QFT the most successful approach to computing multi-point functions of local operators is by expressing them in terms of form factors of individual fields.

Form factors of local fields in integrable QFT can generally be computed exactly by pursuing the so-called form factor programme [38, 39] which was extended to the treatment of branch point twist fields in [24]. The programme has been carried out for countless models and fields and provides extremely accurate results for correlators, particularly two-point functions. Here we are interested in the correlator (2) in the IMMF. Let  $(a, j)$  represent the quantum numbers of a particle of type  $a = 1, \dots, 8$  living in copy  $j = 1, \dots, n$  (we will specialize to  $n = 2$  later on). We may employ the so-called cumulant expansion [40, 41, 42]:

$$\log \left( \frac{\langle \mathcal{T}(0) \tilde{\mathcal{T}}(\ell) \rangle_n}{\langle \mathcal{T} \rangle_n^2} \right) = \sum_{k=1}^{\infty} \frac{c_k(\ell)}{k! (2\pi)^k}, \quad (10)$$

with

$$c_k(\ell) = \sum_{a_1, \dots, a_k=1}^8 \sum_{j_1, \dots, j_k=1}^2 \int_{-\infty}^{\infty} d\theta_1 \cdots \int_{-\infty}^{\infty} d\theta_k h_{j_1 \dots j_k}^{a_1 \dots a_k}(\theta_1, \dots, \theta_k; n) e^{-\ell \sum_{i=1}^k m_{a_i} \cosh \theta_i}, \quad (11)$$

where the functions  $h_{j_1 \dots j_k}^{a_1 \dots a_k}(\theta_1, \dots, \theta_k; n)$  are given in terms of the form factors of the fields involved. In particular,

$$\begin{aligned} h_j^a(\theta; n) &= |F_1^{\mathcal{T}|(a,j)}(\theta; n)|^2 \\ h_{j_1 j_2}^{a_1 a_2}(\theta_1, \theta_2; n) &= F_2^{\mathcal{T}|(a_1, j_1)(a_2, j_2)}(\theta_1, \theta_2; n) (F_2^{\tilde{\mathcal{T}}|(a_1, j_1)(a_2, j_2)}(\theta_1, \theta_2; n))^* - h_{j_1}^{a_1}(\theta_1) h_{j_2}^{a_2}(\theta_2) \end{aligned} \quad (12)$$

where

$$F_1^{\mathcal{T}|(a,j)}(\theta; n) := \frac{\langle 0 | \mathcal{T}(0) | \theta \rangle_{(a,j)}}{\langle \mathcal{T} \rangle_n}, \quad F_2^{\mathcal{T}|(a_1, j_1)(a_2, j_2)}(\theta_1, \theta_2; n) := \frac{\langle 0 | \mathcal{T}(0) | \theta_1 \theta_2 \rangle_{(a_1, j_1)(a_2, j_2)}}{\langle \mathcal{T} \rangle_n} \quad (13)$$

are the normalized one- and two-particle form factors. Here  $\langle 0 |$  represents the vacuum state and  $|\theta\rangle_{(a,j)}$ ,  $|\theta_1 \theta_2\rangle_{(a_1, j_1)(a_2, j_2)}$  represent in-states of 1 and 2 particles, respectively. The states are characterized by the rapidities  $\theta_i$  and particle quantum numbers  $(a_i, j_i)$ . In this paper we will only be interested in one and two particle form factor contributions which provide the most important contributions to (11) for large  $\ell$ . For spinless operators in relativistic theories we have that the one-particle form factors are rapidity independent. Therefore, from now on we will simply write  $F_1^{\mathcal{T}|(a,j)}(\theta; n) := F_1^{\mathcal{T}|(a,j)}(n)$ .

## 2.2 One- and Two-Particle form Factors of the IMMF

Form factors of the IMMF were constructed in [37] for the stress-energy tensor (e.g. the spin field) and in [43] for the energy field. This construction can be easily adapted to the twist field by employing the programme proposed in [24]. In addition, here we only want to study a subset of the one- and two-particle form factors of the twist field which means that we do not need to engage into solving the complicated recursive equations that arise for higher particle numbers. We will also avoid the consideration of higher order poles which has been extensively discussed in [36, 37]. A particular feature of the computation of form factors of twist fields is that many of the basic formulae are very similar to those used in the construction of form factors of standard

local fields, particularly when it comes to constructing the two-particle minimal form factor and therefore, what follows is strongly guided by the analysis of [37].

A minimal solution to the two-particle form factor equation will be denoted by

$$f_{a_1 a_2}(\theta_1 - \theta_2; n) := F_{\min}^{\mathcal{T}|(a_1, j_1)(a_2, j_2)}(\theta_1, \theta_2; n) \quad \text{with } j = 1, \dots, n. \quad (14)$$

This minimal form factor satisfies a twist field version of Watson's equations which may be summarised as [24]

$$f_{a_1 a_2}(\theta, n) = f_{a_1 a_2}(-\theta, n) S_{a_1 a_2}(\theta) = f_{a_1 a_2}(-\theta + 2\pi n i, n). \quad (15)$$

Let  $S_{a_1 a_2}(\theta)$  have an integral representation of the form

$$S_{a_1 a_2}(\theta) = \exp \left[ \int_0^\infty \frac{dt}{t} s_{a_1 a_2}(t) \sinh \left( \frac{t\theta}{i\pi} \right) \right], \quad (16)$$

where  $s_{a_1 a_2}(\theta)$  is a function which depends of the theory and the particles  $a_1, a_2$ . Employing this integral representation it is easy to show that the solution to (15) may be written as

$$f_{a_1 a_2}(\theta; n) = \exp \left[ \int_0^\infty \frac{dt}{t \sinh(nt)} s_{a_1 a_2}(t) \sin^2 \left( \frac{it}{2} \left( n + \frac{i\theta}{\pi} \right) \right) \right]. \quad (17)$$

It is also well-known that if  $S_{a_1 a_2}(\theta) = -1$  in (15) then the free Fermion solution  $f_{a_1 a_2}(\theta) = -i \sinh \frac{\theta}{2n}$  is obtained. Combining these two results and comparing with the formulae provided in [37] for the minimal form factors of the IMMF we find

$$f_{a_1 a_2}(\theta; n) = (-i \sinh \frac{\theta}{2n})^{\delta_{a_1 a_2}} \exp \left[ \sum_{\alpha \in \mathcal{S}_{a_1 a_2}} 2p_\alpha \int_0^\infty dt \frac{\cosh((\alpha - \frac{1}{2})t)}{t \cosh \frac{t}{2} \sinh(nt)} \sin^2 \left( \frac{it}{2} \left( n + \frac{i\theta}{\pi} \right) \right) \right], \quad (18)$$

where the exponential may be also expressed as the infinite product of gamma functions:

$$\prod_{\alpha \in \mathcal{S}_{a_1 a_2}} \prod_{k=0}^{\infty} \left[ \frac{\Gamma \left( \frac{k+n-\alpha+1}{2n} \right)^2 \Gamma \left( \frac{k+n+\alpha}{2n} \right)^2}{\Gamma \left( \frac{k-\alpha-\frac{i\theta}{\pi}+1}{2n} \right) \Gamma \left( \frac{k+\alpha-\frac{i\theta}{\pi}}{2n} \right) \Gamma \left( 1 + \frac{k-\alpha+\frac{i\theta}{\pi}+1}{2n} \right) \Gamma \left( 1 + \frac{k+\alpha+\frac{i\theta}{\pi}}{2n} \right)} \right]^{p_\alpha (-1)^k} \quad (19)$$

and the set  $\mathcal{S}_{a_1 a_2}$  has been defined after (5). Following [24] and [37], the most generic two-particle form factor takes the form

$$F_2^{\mathcal{T}|(a_1, j_1)(a_2, j_2)}(\theta_1, \theta_2; n) = \frac{Q_{a_1 a_2}^{j_1 j_2}(\theta_{12}; n)}{2n K_{j_1 j_2}(\theta_{12}; n)^{\delta_{a_1 a_2}} \prod_{\alpha \in \mathcal{S}_{a_1 a_2}} (B_\alpha(\theta_{12}; n)^{u(p_\alpha)} B_{1-\alpha}(\theta_{12}; n)^{v(p_\alpha)})^{\delta_{j_1 j_2}}} \times \frac{F_{\min}^{\mathcal{T}|(a_1, j_1)(a_2, j_2)}(\theta_1, \theta_2; n)}{F_{\min}^{\mathcal{T}|(a_1, j_1)(a_2, j_2)}(i\pi, 0; n)}, \quad (20)$$

with  $\theta_{12} := \theta_1 - \theta_2$ ,

$$K_{j_1 j_2}(\theta; n) = \frac{\sinh\left(\frac{i\pi(1-2(j_1-j_2))-\theta}{2n}\right) \sinh\left(\frac{i\pi(1-2(j_1-j_2))+\theta}{2n}\right)}{\sin \frac{\pi}{n}}, \quad (21)$$

$$B_\alpha(\theta; n) = \sinh\left(\frac{i\pi\alpha - \theta}{2n}\right) \sinh\left(\frac{i\pi\alpha + \theta}{2n}\right), \quad (22)$$

and

$$u(2k+1) = k+1, \quad u(2k) = k \quad \text{and} \quad v(2k+1) = v(2k) = k \quad \text{for } k \in \mathbb{Z}. \quad (23)$$

The function  $K_{j_1 j_2}(\theta; n)$  encodes the full kinematic pole structure of the form factor, having kinematic poles at  $\theta = i\pi$  and  $\theta = i\pi(2n-1)$  in the extended physical strip  $\text{Im}(\theta) \in [0, 2\pi n]$ , whereas  $B_\alpha(\theta; n)$  encodes the bound state pole structure, as characterised in [37]. The minimal form factors in (20) can be easily obtained from (18) by employing standard relations which can be found for instance in [24]. Finally, the functions  $Q_{a_1 a_2}^{j_1 j_2}(\theta; n)$  are solutions to the equations:

$$Q_{a_1 a_2}^{11}(\theta; n) = Q_{a_1 a_2}^{11}(-\theta; n) = Q_{a_1 a_2}^{11}(-\theta + 2\pi i n; n) \quad (24)$$

namely, they are linear combinations of functions of the type  $\cosh^k \frac{\theta}{n}$  for  $k = 1, 2, \dots$ , similar to the ansatz already employed in [37]. Which values of  $k$  are involved will be determined by constraints on how  $Q_{a_1 a_2}^{11}(\theta; n)$  behaves as  $\theta \rightarrow \infty$  which we will discuss in the next subsection.

In this paper we will only require the two-particle form factors  $F_2^{(1,1)(1,1)}(\theta_1, \theta_2; n)$  and  $F_2^{(1,1)(2,1)}(\theta_1, \theta_2; n)$ . These are special cases of (20) explicitly given by

$$F_2^{\mathcal{T}[(1,1)(1,1)]}(\theta_1, \theta_2; n) = \frac{Q_{11}^{11}(\theta_{12}; n)}{2n K_{11}(\theta_{12}; n)} \prod_{\alpha=\frac{2}{3}, \frac{2}{5}, \frac{1}{15}} B_\alpha(\theta_{12}; n) \frac{f_{11}(\theta_{12}; n)}{f_{11}(i\pi; n)}, \quad (25)$$

and

$$F_2^{\mathcal{T}[(1,1)(2,1)]}(\theta_1, \theta_2; n) = \frac{Q_{12}^{11}(\theta_{12}; n)}{2n} \prod_{\alpha=\frac{4}{5}, \frac{3}{5}, \frac{7}{15}, \frac{4}{15}} B_\alpha(\theta_{12}; n) \frac{f_{12}(\theta_{12}; n)}{f_{12}(i\pi; n)}, \quad (26)$$

with

$$f_{11}(\theta; n) = -i \sinh \frac{\theta}{2n} \exp \left[ 2 \int_0^\infty \frac{dt}{t} \frac{\cosh \frac{t}{10} + \cosh \frac{t}{6} + \cosh \frac{13t}{30}}{\cosh \frac{t}{2} \sinh(nt)} \sin^2 \left( \frac{it}{2} \left( n + \frac{i\theta}{\pi} \right) \right) \right], \quad (27)$$

and

$$f_{12}(\theta; n) = \exp \left[ 2 \int_0^\infty \frac{dt}{t} \frac{\cosh \frac{t}{10} + \cosh \frac{3t}{10} + \cosh \frac{t}{30} + \cosh \frac{7t}{30}}{\cosh \frac{t}{2} \sinh(nt)} \sin^2 \left( \frac{it}{2} \left( n + \frac{i\theta}{\pi} \right) \right) \right]. \quad (28)$$



### 2.3 Fixing One- and Two-Particle Form Factors

The equations (25)-(26) give the two-particle form factors of interest up to the functions  $Q_{11}^{11}(\theta; n)$  and  $Q_{12}^{11}(\theta; n)$ . As anticipated earlier, these functions may be determined by employing additional constraints. In particular, the kinematic and bound state residue equations for the two-particle form factors require that:

$$\lim_{\bar{\theta}_0 \rightarrow \theta_0} (\theta_0 - \bar{\theta}_0) F_2^{\mathcal{T}|(a,j)(\bar{a},j)}(\bar{\theta}_0 + i\pi, \theta_0; n) = i, \quad (29)$$

and

$$\lim_{\bar{\theta}_0 \rightarrow \theta_0} (\theta_0 - \bar{\theta}_0) F_2^{\mathcal{T}|(a_1,j)(a_2,j)}(\bar{\theta}_0 + \frac{i\pi\alpha}{2}, \theta_0 - \frac{i\pi\alpha}{2}; n) = i\Gamma_{a_1 a_2}^{a_3} F_1^{\mathcal{T}|(a_3,j)}(n). \quad (30)$$

In the IMMF all particles are self conjugate so that the form factor (25) must satisfy the condition (29), giving the constraint:

$$Q_{11}^{11}(i\pi; n) = \prod_{\alpha=5,9,14,16,21,25,30} \sin \frac{\pi\alpha}{30n}. \quad (31)$$

The same form factor possesses three bound state poles related to the formation of bound states, 1, 2 and 3. This means that it satisfies three versions of equation (30), giving three additional constraints:

$$Q_{11}^{11}\left(\frac{2\pi i}{3}; n\right) = -\Gamma_{11}^1 F_1^{\mathcal{T}|(1,1)}(n) \csc \frac{\pi}{n} \frac{f_{11}(i\pi; n)}{f_{11}\left(\frac{2\pi i}{3}; n\right)} \prod_{\alpha=4,5,9,11,16,20,25} \sin \frac{\alpha\pi}{30n}, \quad (32)$$

$$Q_{11}^{11}\left(\frac{2\pi i}{5}; n\right) = \Gamma_{11}^2 F_1^{\mathcal{T}|(2,1)}(n) \csc \frac{\pi}{n} \frac{f_{11}(i\pi; n)}{f_{11}\left(\frac{2\pi i}{5}; n\right)} \prod_{\alpha=4,5,7,9,12,16,21} \sin \frac{\alpha\pi}{30n}, \quad (33)$$

$$Q_{11}^{11}\left(\frac{i\pi}{15}; n\right) = -\Gamma_{11}^3 F_1^{\mathcal{T}|(3,1)}(n) \csc \frac{\pi}{n} \frac{f_{11}(i\pi; n)}{f_{11}\left(\frac{i\pi}{15}; n\right)} \prod_{\alpha=2,5,7,9,11,14,16} \sin \frac{\alpha\pi}{30n}. \quad (34)$$

The form factor (26) has no kinematic poles but has four bound state poles associated to the formation of bound states 1, 2, 3 and 4. They give the additional constraints:

$$Q_{12}^{11}\left(\frac{4\pi i}{5}; n\right) = \Gamma_{12}^1 F_1^{\mathcal{T}|(1,1)}(n) \frac{f_{12}(i\pi; n)}{f_{12}\left(\frac{4\pi i}{5}; n\right)} \prod_{\alpha=3,5,8,16,19,21,24} \sin \frac{\alpha\pi}{30n}, \quad (35)$$

$$Q_{12}^{11}\left(\frac{3\pi i}{5}; n\right) = -\Gamma_{12}^2 F_1^{\mathcal{T}|(2,1)}(n) \frac{f_{12}(i\pi; n)}{f_{12}\left(\frac{3\pi i}{5}; n\right)} \prod_{\alpha=2,3,5,13,16,18,21} \sin \frac{\alpha\pi}{30n}, \quad (36)$$

$$Q_{12}^{11}\left(\frac{7i\pi}{15}; n\right) = \Gamma_{12}^3 F_1^{\mathcal{T}|(3,1)}(n) \frac{f_{12}(i\pi; n)}{f_{12}\left(\frac{7i\pi}{15}; n\right)} \prod_{\alpha=2,3,5,11,14,16,19} \sin \frac{\alpha\pi}{30n}, \quad (37)$$

and

$$Q_{12}^{11}\left(\frac{4i\pi}{15}; n\right) = -\Gamma_{12}^4 F_1^{\mathcal{T}|(4,1)}(n) \frac{f_{12}(i\pi; n)}{f_{12}\left(\frac{4i\pi}{15}; n\right)} \prod_{\alpha=3,5,8,11,13,16} \sin \frac{\alpha\pi}{30n}. \quad (38)$$

At this stage we have obtained 8 equations and have 4 unknowns, corresponding to the one particle form factors  $F_1^{\mathcal{T}|(a,1)}(n)$  with  $a = 1, 2, 3, 4$  as well as the polynomials  $Q_{11}^{11}(\theta; n)$  and  $Q_{12}^{11}(\theta; n)$ . We know from the two particle form factor equations that they must be even functions of  $\theta$  and we would also like to require the cluster decomposition property which has been discussed in detail in [47] and observed for numerous models (a particularly rich example can be found here [48]), namely that

$$\lim_{\theta_1 \rightarrow \infty} F_2^{\mathcal{T}|(1,1)(1,1)}(\theta_1, \theta_2; n) = (F_1^{\mathcal{T}|(1,1)}(n))^2. \quad (39)$$

and

$$\lim_{\theta_1 \rightarrow \infty} F_2^{\mathcal{T}|(1,1)(2,1)}(\theta_1, \theta_2; n) = F_1^{\mathcal{T}|(1,1)}(n) F_1^{\mathcal{T}|(2,1)}(n). \quad (40)$$

These properties provide very strong constraints for the functions  $Q_{11}^{11}(\theta; n)$  and  $Q_{12}^{11}(\theta; n)$ , as they allow us to determine the highest power of  $e^{\theta/n}$  that can be involved. We have that

$$K_{11}(\theta; n) \sim -\frac{1}{4}e^{\frac{\theta}{n}} \quad \text{and} \quad B_\alpha(\theta; n) \sim -\frac{1}{4}e^{\frac{\theta}{n}} \quad \text{for} \quad \theta \rightarrow \infty. \quad (41)$$

It is also easy to determine the leading behaviours of  $f_{11}(\theta; n)$  and  $f_{12}(\theta; n)$  as  $\theta \rightarrow \infty$ . In this limit, integrals of the type

$$\exp \left[ 2 \int_0^\infty dt \frac{\cosh((\alpha - \frac{1}{2})t)}{t \cosh \frac{t}{2} \sinh(nt)} \sin^2 \left( \frac{it}{2} \left( n + \frac{i\theta}{\pi} \right) \right) \right], \quad (42)$$

may be approximated by changing variables to  $x = t\theta$  and then expanding for small values of  $\frac{x}{\theta}$ . At leading and next-to-leading order, this yields the simple integral:

$$\exp \left[ \int_0^\infty dx \left( \frac{2\theta}{nx^2} \sin^2 \left( \frac{x}{2\pi} \right) - \frac{i}{x} \sin \left( \frac{x}{\pi} \right) \right) \right] = -ie^{\frac{\theta}{2n}}. \quad (43)$$

Numerical evaluation of the minimal form factors for  $\theta$  large confirms the behaviour above, up to  $n$ -dependent proportionality constants which we, unfortunately, have been unable to find a convergent analytic expression for

$$f_{11}(\theta; n) \sim v(n)e^{\frac{2\theta}{n}}, \quad f_{12}(\theta; n) \sim u(n)e^{\frac{2\theta}{n}} \quad \text{for} \quad \theta \rightarrow \infty. \quad (44)$$

Nonetheless, the numbers  $v(n)$  and  $u(n)$  can be numerically estimated for every  $n$ . This means that, in order to satisfy (39)-(40) we need

$$Q_{11}^{11}(\theta; n) \sim e^{\frac{2\theta}{n}} \quad \text{and} \quad Q_{12}^{11}(\theta; n) \sim e^{\frac{2\theta}{n}} \quad \text{for} \quad \theta \rightarrow \infty. \quad (45)$$

thus, in general

$$Q_{11}^{11}(\theta; n) = A_{11}^{11}(n) + B_{11}^{11}(n) \cosh \frac{\theta}{n} + C_{11}^{11}(n) \cosh^2 \frac{\theta}{n}, \quad (46)$$

and

$$Q_{12}^{11}(\theta; n) = A_{12}^{11}(n) + B_{12}^{11}(n) \cosh \frac{\theta}{n} + C_{12}^{11}(n) \cosh^2 \frac{\theta}{n}. \quad (47)$$

And the cluster decomposition equations (39)-(40) can be written as

$$(F_1^{\mathcal{T}|(1,1)}(n))^2 = \frac{4^3 \sin \frac{\pi}{n} C_{11}^{11}(n) v(n)}{2n f_{11}(i\pi; n)} \quad \text{and} \quad F_1^{\mathcal{T}|(2,1)}(n) F_1^{\mathcal{T}|(1,1)}(n) = \frac{4^3 C_{12}^{11}(n) u(n)}{2n f_{12}(i\pi; n)}. \quad (48)$$

It is worth noting that the same conclusions regarding the form of the functions  $Q_{11}^{11}(\theta; n)$  and  $Q_{12}^{11}(\theta; n)$  can be reached by appealing to a well-known argument presented for instance in [37] according to which the form factors of unitary operators (in particular, the twist field) can diverge at most as  $e^{\Delta_n \theta_i}$  when one of the rapidities  $\theta_i \rightarrow \infty$  and where  $\Delta_n$  is the conformal dimension (3) of the twist field. In fact, for  $c = \frac{1}{2}$  it turns out that  $\Delta_n < 1$  for  $n < 49$  and therefore, at least for a wide range of values of  $n$  the form factors must tend to a constant as any of the rapidities they depend upon tends to infinity. The property of cluster decomposition, additionally establishes what this constant must be.

Putting together equations (31)-(38) and (48) we end up with 10 equations for 10 unknowns:  $A_{11}^{11}(n), B_{11}^{11}(n), C_{11}^{11}(n), A_{12}^{11}(n), B_{12}^{11}(n), C_{12}^{11}(n)$  and the four one particle form factors  $F_1^{\mathcal{T}|(1,1)}(n), F_1^{\mathcal{T}|(2,1)}(n), F_1^{\mathcal{T}|(3,1)}(n)$  and  $F_1^{\mathcal{T}|(4,1)}(n)$ . This means we are now in a position to determine all these functions and to investigate how our results apply to the study of the second Rényi entropy.

### 3 Second Rényi Entropy of the IMMF

From the definition (2) and the expansion (10) we know that the Rényi entropy may be expressed as an infinite sum involving integrals over the form factors of the twist field. In particular the first few contributions can be written as

$$\begin{aligned} S_2(\ell) - S_2(\infty) &= -\frac{2}{\pi} \sum_{a=1}^8 |F_1^{\mathcal{T}|(a,1)}(2)|^2 K_0(m_a \ell) \\ &\quad - \frac{1}{(2\pi)^2} \sum_{a_1, a_2=1}^8 \sum_{j=1}^2 \int_{-\infty}^{\infty} d\theta_1 \int_{-\infty}^{\infty} d\theta_2 h_{a_1 a_2}^{1j}(\theta_1, \theta_2; 2) e^{-m_{a_1} \ell \cosh \theta_1 - m_{a_2} \ell \cosh \theta_2} + \dots \end{aligned} \quad (49)$$

where

$$h_{a_1 a_2}^{1j}(\theta_1, \theta_2; 2) := |F_2^{\mathcal{T}|(a_1,1)(a_2,j)}(\theta_1, \theta_2; 2)|^2 - |F_1^{\mathcal{T}|(a_1,1)}(2)|^2 |F_1^{\mathcal{T}|(a_2,j)}(2)|^2. \quad (50)$$

and  $S_2(\infty) = -4\Delta_2 \log(m\epsilon) - 2 \log \langle \mathcal{T} \rangle_2$  is the non-universal saturation constant. The expansion above describes corrections to saturation which are exponentially decaying for large  $m_1 \ell$ . If we consider all terms above, it becomes quickly apparent that some of the one-particle form factor contributions are subleading compared to some of the two-particle form factor contributions, for  $\ell$  large. This is because, as mentioned earlier in the paper, the masses of particles  $4, \dots, 8$  are all larger than twice the mass of particle 1. In summary, this means that, the first six leading

form-factor corrections to saturation in order of importance are

$$\begin{aligned}
S_2(\ell) - S_2(\infty) &= -\frac{2}{\pi} \sum_{a=1}^3 |F_1^{\mathcal{T}|(a,1)}(2)|^2 K_0(m_a \ell) \\
&- \frac{1}{2\pi^2} \sum_{j=1}^2 \int_{-\infty}^{\infty} d\theta h_{11}^{1j}(\theta, 0; 2) K_0(2m_1 \ell \cosh \theta/2) - \frac{2}{\pi} |F_1^{\mathcal{T}|(4,1)}(2)|^2 K_0(m_4 \ell) \\
&- \frac{1}{2\pi^2} \sum_{j=1}^2 \int_{-\infty}^{\infty} d\theta h_{12}^{1j}(\theta, 0; 2) K_0(\ell \sqrt{m_1^2 + m_2^2 + 2m_1 m_2 \cosh \theta}) - \dots \quad (51)
\end{aligned}$$

where

$$\begin{aligned}
&\sum_{j=1}^2 h_{1a}^{1j}(\theta, 0; 2) = \\
&|F_2^{\mathcal{T}|(1,1)(a,1)}(\theta, 0; 2)|^2 + |F_2^{\mathcal{T}|(1,1)(a,1)}(-\theta + 2\pi i, 0; 2)|^2 - 2|F_1^{\mathcal{T}|(1,1)}(2)|^2 |F_1^{\mathcal{T}|(a,1)}(2)|^2, \quad (52)
\end{aligned}$$

and we have carried out one of the integrals in (49). For  $m_1 \ell \gg 1$  the leading contribution to the first integral in (51) can be written as:

$$\frac{1}{2\pi^2} \sum_{j=1}^2 \int_{-\infty}^{\infty} d\theta h_{11}^{1j}(\theta, 0; 2) K_0(2m_1 \ell \cosh \theta/2) \approx \sqrt{\frac{\pi}{m_1 \ell}} \frac{e^{-2m_1 \ell}}{4\pi^2} \sum_{j=1}^2 \int_{-\infty}^{\infty} d\theta \frac{h_{11}^{1j}(\theta, 0; 2)}{\sqrt{\cosh \frac{\theta}{2}}}, \quad (53)$$

and similarly for the last integral

$$\begin{aligned}
&\frac{1}{2\pi^2} \sum_{j=1}^2 \int_{-\infty}^{\infty} d\theta h_{12}^{1j}(\theta, 0; 2) K_0(\ell \sqrt{m_1^2 + m_2^2 + 2m_1 m_2 \cosh \theta}) \approx \\
&\sqrt{\frac{\pi}{2m_1 \ell}} \frac{e^{-(m_1+m_2)\ell}}{2\pi^2} \sum_{j=1}^2 \int_{-\infty}^{\infty} d\theta \frac{h_{12}^{1j}(\theta, 0; 2)}{\sqrt[4]{1 + \frac{m_2^2}{m_1^2} + 2\frac{m_2}{m_1} \cosh \theta}}, \quad (54)
\end{aligned}$$

where the remaining integrals are convergent and can be easily evaluated numerically.

### 3.1 Numerical Computation of the One-Particle Form Factors

Although we are now in a position to solve equations (31)-(38) and (39)-(40) and therefore obtain explicit formulae for the one- and two-particle form factors of interest, in practice these equations are rather cumbersome and finding exact formulae is extremely difficult (even for  $n = 2$ ). They can however be very easily solved numerically for any given value of  $n$ .

An interesting observation from solving the equations numerically is that the solution is not unique. This is mainly due to the equations (39)-(40) which are quadratic in the one-particle form factors. There are in fact three solutions for each value of  $n$ . This obviously poses the question as to which of these solutions is the one we are looking for. It also indicates that the

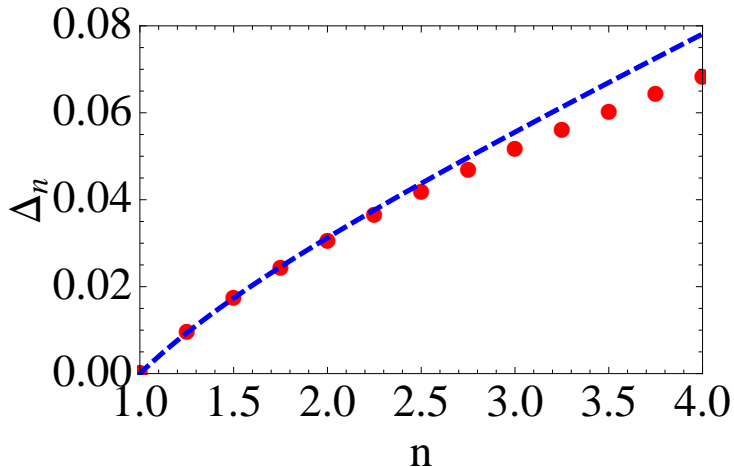


Figure 3: The function  $\Delta_n = \frac{1}{48} \left( n - \frac{1}{n} \right)$  (dashed line) compared to values of  $\Delta_n$  obtained by evaluating the  $\Delta$ -sum rule (circles) including the first 6 leading contributions to the form factor expansion (similar to (51)).

form factor equations allow for several twist field solutions. This is not surprising as all operators enjoying the twist-property should have form factors satisfying the same set of equations. Among those operators there is the branch point twist field we are interested in, but also other twist fields, such as the composite twist fields introduced in [44] and studied in [45, 21, 46]. These are fields which are defined (at criticality) as the leading field in the OPE of the standard branch point twist field and any other local field of the replica theory.

Fortunately, there is one simple way of telling such composite twist fields and the twist fields we are interested in apart: composite twist fields have form factors which do not vanish at  $n = 1$  whereas the twist field  $\mathcal{T}$  must reduce to the identity field at  $n = 1$ , hence all its form factors vanish at  $n = 1$ . Imposing this condition we find a single solution with the desired property of having vanishing form factors at  $n = 1$ . In addition, there are certain consistency checks that we may further apply to test this solution. A common test is the  $\Delta$ -sum rule proposed in [47] which may be written as

$$\Delta_n = -\frac{1}{2\langle \mathcal{T} \rangle_n} \int_0^\infty dr r \langle \Theta(r) \mathcal{T}(0) \rangle_n, \quad (55)$$

where  $\Theta(r)$  is the energy-momentum tensor. Employing a standard form factor expansion, it is then possible to recover the conformal dimension of the twist field (3) from its two-point function with the energy-momentum tensor. This computation is possible thanks once more to the results of [37] where the form factors of the energy-momentum tensor were obtained, in particular all one- and two-particle form factors. Fig. 3 shows the numerically obtained values of  $\Delta_n$  employing (55). As can be seen it is possible to compute these values also for non-integer  $n$  as the form factors are well-defined for all values of  $n$ . It is also noticeable that the saturation of the  $\Delta$ -sum rule becomes worse the larger  $n$  is. This is a rather common feature of the

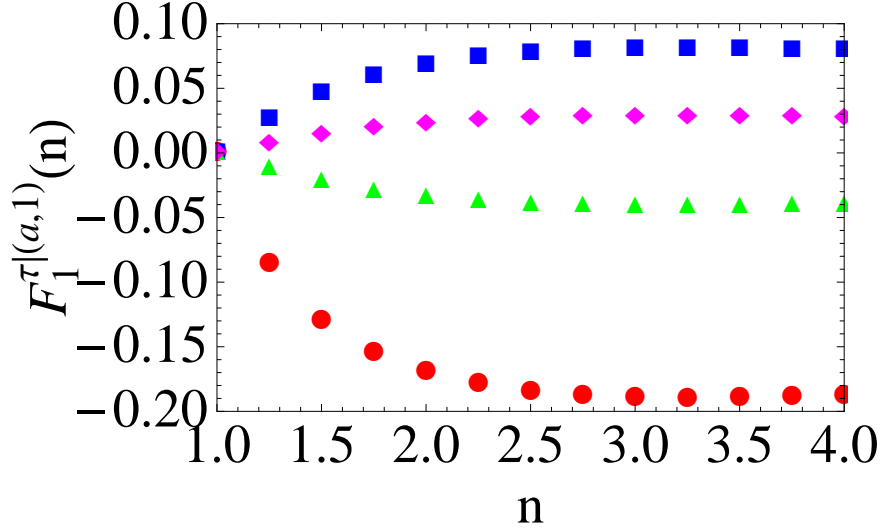


Figure 4: One particle form factors associated to particle 1 (circles), particle 2 (squares), particle 3 (triangles) and particle 4 (rombi).

branch point twist which is due to the fact that all form factor contributions to the expansion are proportional to  $n$  and so the weight of further form factor corrections is increased as  $n$  increases. We have also observed that the numerical determination of the constants  $u(n)$  and  $v(n)$  in the asymptotics (44) becomes more difficult for larger  $n$ . Both constants are the result of the numerical integration of a decaying but wildly oscillating function which is delicate. A less precise knowledge of the values  $u(n)$  and  $v(n)$  may well also contribute to the worsening agreement observed in Fig. 3. Fortunately though we have very good agreement for  $n = 2$  so that we can be confident about the corresponding form factors.

The first four one-particle form factors  $F_1^{\tau|(a,1)}(n)$  for  $a = 1, 2, 3, 4$  are all real numbers and their values are presented in Fig. 4. All values are rather small and, in absolute value, are smaller the higher the particle label. In particular:

$$\begin{aligned} F_1^{\tau|(1,1)}(2) &= -0.169286, & F_1^{\tau|(2,1)}(2) &= 0.0687845, \\ F_1^{\tau|(3,1)}(2) &= -0.0336516 & \text{and} & F_1^{\tau|(4,1)}(2) = 0.0230515. \end{aligned} \quad (56)$$

We can also numerically determine the values of the coefficients  $A_{11}^{1a}(n)$ ,  $B_{1a}^{11}(n)$  and  $C_{1a}^{11}(n)$  for  $a = 1, 2$ . For  $n = 2$  they are:

$$\begin{aligned} A_{11}^{11}(2) &= 0.0502866, & A_{12}^{11}(2) &= -0.0387777, \\ B_{11}^{11}(2) &= 0.0000930, & B_{12}^{11}(2) &= 0.0002876, \\ C_{11}^{11}(2) &= 0.0091876, & C_{12}^{11}(2) &= -0.0043528. \end{aligned} \quad (57)$$

With these values, it is now possible to evaluate the integrals (53)-(54):

$$\int_{-\infty}^{\infty} d\theta \frac{h_{11}^{11}(\theta, 0; 2) + h_{11}^{12}(\theta, 0; 2)}{\sqrt{\cosh \frac{\theta}{2}}} = 0.100857, \quad \int_{-\infty}^{\infty} d\theta \frac{h_{12}^{11}(\theta, 0; 2) + h_{12}^{12}(\theta, 0; 2)}{\sqrt[4]{1 + \frac{m_2^2}{m_1^2} + 2\frac{m_2}{m_1} \cosh \theta}} = 0.0181714. \quad (58)$$

### 3.2 Exact Corrections to Saturation

Putting together all the numerical values found in the previous section, we see that the formula (51) may be expressed as:

$$\begin{aligned} S_2(\ell) - S_2(\infty) &= -0.0182441K_0(m_1\ell) - 0.00301205K_0(m_2\ell) \\ &\quad - 0.000720926K_0(m_3\ell) - \frac{1}{2\pi^2} \sum_{j=1}^2 \int_{-\infty}^{\infty} d\theta h_{11}^{1j}(\theta, 0; 2) K_0(2m_1\ell \cosh \theta/2) \\ &\quad - 0.000338282K_0(m_4\ell) \\ &\quad - \frac{1}{2\pi^2} \sum_{j=1}^2 \int_{-\infty}^{\infty} d\theta h_{12}^{1j}(\theta, 0; 2) K_0(\ell \sqrt{m_1^2 + m_2^2 + 2m_1m_2 \cosh \theta}). \end{aligned} \quad (59)$$

where the leading contributions to the integrals are  $0.00452814 \frac{e^{-2m_1\ell}}{\sqrt{m_1\ell}}$  and  $0.00115377 \frac{e^{-(m_1+m_2)\ell}}{\sqrt{m_1\ell}}$ , respectively. From (59) it is clear that the one-particle form factor contributions are led by small coefficients (as the one-particle form factors are all small numbers). However, the two-particle contributions are characterized by even smaller coefficients and are in fact strongly suppressed (they are essentially negligible for  $m_1\ell$  above 1), as shown in Fig. 5. Therefore the hypothesis

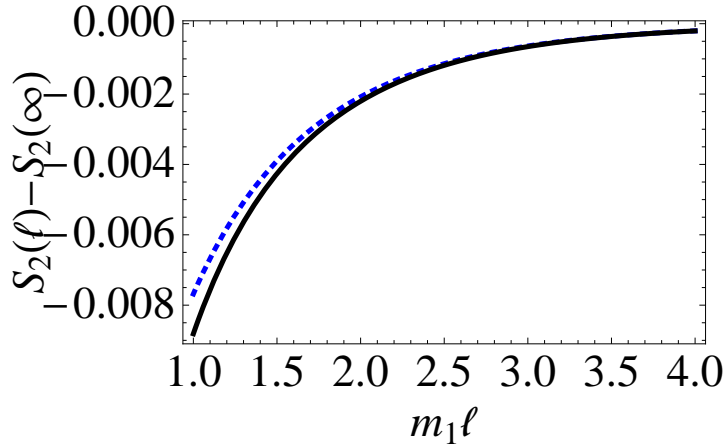


Figure 5: The function (51) (solid curve). The dashed curve is the contribution  $-0.0182441K_0(m_1\ell)$ , that is the leading contribution to (51). Clearly the contribution proportional to  $K_0(m_1\ell)$  is leading for the full range of values of  $\ell$  considered here.

put forward in [7] that the one-particle form factors are too small (perhaps zero) to be detectable numerically seems incorrect. There must therefore be another reason why such one-particle form factor contributions are not visible in TCSA numerics.

## 4 Comparison to TCSA Results

An interesting and surprising result of [7] is that a numerical evaluation of the next-to-leading order correction to the second Rényi entropy is well fitted by the function<sup>1</sup>:

$$f(\ell) = -0.04e^{-2m_1\ell}. \quad (60)$$

From this result and the discussion in [7] it is clear that no evidence of order  $e^{-m_1\ell}$ ,  $e^{-m_2\ell}$  or  $e^{-m_3\ell}$  corrections was found, even though, according to our formula (51) they should not only be present but  $e^{-m_1\ell}$  should be the leading correction to saturation. One would expect that at least this leading correction would be identifiable through numerical approaches such as TCSA. Before discussing the comparison of the TCSA data in Fig. 6 with our formula (59) there is one important question worth raising. Are these two results comparable at all? That is, do they describe the same sort of physics? To answer these questions, let us consider the set-ups in which our and the results of [7] apply. The result (59) follows from a form factor approach

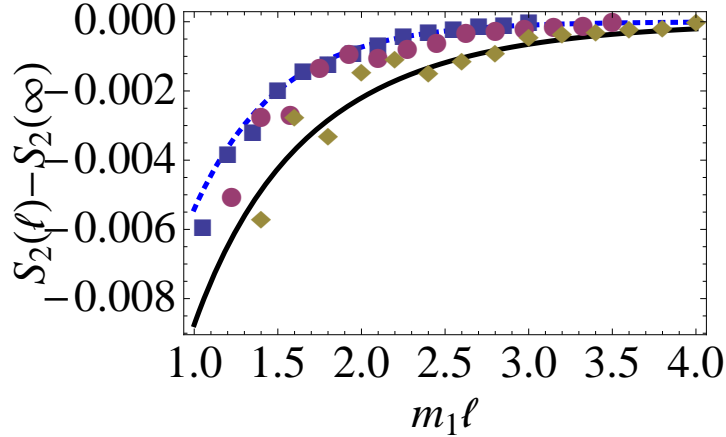


Figure 6: The function (51) (solid curve), compared to TCSA results obtained in [7] for a system of length  $m_1L = 8$  (rombi),  $m_1L = 7$  (circles) and  $m_1L = 6$  (squares) and to the fit (60) (dashed curve).

and describes the entanglement of a finite subsystem of length  $\ell$  within an *infinite* system. In contrast, the results of [7] measure the entanglement of a subsystem of length  $\ell$  within a *finite* system of length  $L$  and the values of  $m_1\ell$  considered run between 0 and  $m_1L/2$ . Given that

<sup>1</sup>I would like to thank Tamás Pálmai for sharing with me the numerical data in Fig. 6, previously published in [7].



$m_1 L = 8$  is the largest system that is considered in the work [7] the question naturally arises as to whether or not it makes sense to compare results for such a system to those in infinite volume. The answer appears to be yes, the results are comparable. The reason for this is that although  $m_1 L = 8$  may not seem a large value, it has been observed numerically that higher values produce comparable results. In other words,  $m_1 L = 8$  is already very close to infinite volume. Evidence for this conclusion can also be gathered through other computations, such as those of the energy spectrum and the entanglement entropy of simpler theories. If we accept that TCSA results are close to infinite volume results we must also conclude that boundary effects due to the fact that the TCSA results are obtained for a closed system (rather than infinite and open, as for the form factors) should also be negligible, especially once the saturation constant has been subtracted. Under these two assumptions we should expect that the form factor and TCSA results are comparable.

A comparison of the results obtained by the two approaches is provided in Fig. 6. Considering this figure one would have to conclude that both the form factor result (59) and the fit (60) provide a reasonably good description of the numerical data, even though they are very different functions with *different* leading exponential decay. The question is then, which of two curves provides a better fit of the data and a potential best fit of the infinite volume data. Whereas (60) fits the data with  $m_1 L = 6$  very well, the form factors seem to provide a better fit of the data for the larger system  $m_1 L = 8$ . These data on the other hand seem less precise (e.g. there are some oscillations due to numerical inaccuracy). However, it seems clear from the whole set of data (and it is also noted in [7]) that the effect of increasing the volume is to shift the curves slightly down. This is clearly observed for the three values presented in Fig. 6. It is therefore plausible that the solid curve is approached asymptotically as the volume is increased. On the other hand, the fit (60) does not seem to provide a good description as volume is increased.

We conclude from this analysis that it is actually very difficult to correctly identify subleading exponential corrections to entanglement purely from a TCSA analysis, as this requires a very precise understanding of the infinite volume limit. This is particularly difficult when considering the Rényi entropies as the coefficients of the expected exponential decays are model-dependent and not generally known a priori. Therefore, prior to this study, it was not possible to know if the leading term proportional to  $K_0(m_1 \ell)$  was going to be suppressed by, for instance, an extremely small value of the one-particle form factor. One may predict a certain order of magnitude for such form factors, based for instance on consistency checks, such as the  $\Delta$ -sum but it would still be very hard to say anything accurate about the comparison in the order of magnitude of the various terms in (51) or in any other model with a complicated particle content.

## 5 Conclusions and Outlook

In this paper we have carried out an in depth study of the second Rényi entropy in the transverse field Ising model or  $E_8$  minimal Toda field theory employing the branch point twist field approach developed in [24]. This work, follows on from a stream of works where the entanglement of various particular integrable models has been studied [24, 49, 25, 50, 51, 52] by the same method. Our results provide the first detailed form factor study of the entanglement of an interacting theory with a complicated particle content and no internal symmetries so that all twist field form factors

are non-vanishing. We have compared our results to those obtained by a new method for the evaluation of measures of entanglement recently developed by T. Pálmai [7]. This new method is based on the use of the truncated conformal space approach [29] to massive integrable quantum field theories.

The main objective of the work was to explore the level of agreement and complementarity between the branch point twist field and TCSA approaches. A priori, the results of [7] seemed to disagree with a prediction based on the use of branch point twist fields, namely that the Rényi entropy of massive QFT should saturate for large subsystem sizes and acquire exponentially decaying corrections to saturation of order  $e^{-m_1\ell}$  for  $m_1\ell$  sufficiently large,  $m_1$  being the mass of the lightest particle in the spectrum [24]. Instead, the numerical approach employed in [7] initially suggested that the leading correction was instead of the form  $e^{-2m_1\ell}$ .

In this paper we have provided analytic evidence that the solution to this apparent contradiction is simple: exponentially decaying corrections of the form  $e^{-m_1\ell}$  are indeed present and leading, however within the TCSA approach they only become clearly detectable when data are extrapolated to their infinite volume limit. For large but finite volumes, the TCSA data are not precise enough to differentiate between a  $e^{-2m_1\ell}$  and a  $e^{-m_1\ell}$  decay, especially if there is no a priori knowledge of the order of magnitude of the coefficients of such decays. It is worth noting however that given the different nature of the two approaches and the inherent margin of error associated with the numerics [7], the agreements between the numerical data and the form factor data displayed in Fig. 6 is still remarkable.

An interesting byproduct of this research is that it further confirms that numerical and analytical computations of the Rényi entropy in massive QFT may be employed as a means to prove the low energy spectrum of the theory. However, what this particular example shows us is that if do not know the spectrum of the QFT a priori, then the Rényi entropy will exhibit exponentially decaying corrections where the decay rate may difficult to identify with precision. Indeed, as there is no fundamental reason we know of for the coefficients of the various corrections in (59) to be large or small, one may argue that there could be theories where even the decay  $e^{-2m_1\ell}$  is suppressed or where terms involving decaying exponentials depending on the masses of heavier particles are enhanced by large coefficients.

Given the above the von Neumann entropy emerges as a measure of entanglement which can give more reliable information regarding the mass spectrum of the model under consideration. The reason is that we know from [24] that the leading correction to saturation of the von Neumann entropy is universally given by  $-\frac{1}{8}K_0(2m_1\ell)$  with fixed coefficient. It is therefore desirable that the TCSA approach may be extended to dealing directly with the von Neumann entropy.

Overall this work provides evidence that the TCSA approach and the branch point twist field approach complement each other and lead to consistent results. This opens the interesting prospect of employing the TCSA approach to numerically study models which are particularly complicated to analyze from a branch point twist field perspective and to test theoretical predictions based on the use of branch point twist fields.

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